Simulation and Visualization of Optimal Geometry

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Co-Chair, Berlin Mathematical School
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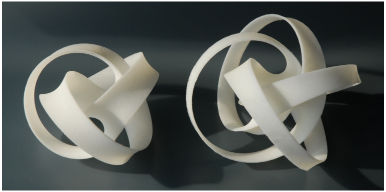
Illustrating Mathematics
ICERM, Providence, 30 June / 1 July 2016



Berlin Mathematical School

- Joint math graduate school
- Funded since 2006 by German *Excellence Initiative*
- Combined offerings of three departments

Berlin Mathematical School

Berlin
Mathematical
School

- Fast-track program (Bachelor to Ph.D. in 4–5 years)
- Basic and advanced graduate courses in English
- Thesis research often within RTG, SFB, etc.
- Scholarships available
- Mentoring, soft-skills, summer schools

`www.math-berlin.de`

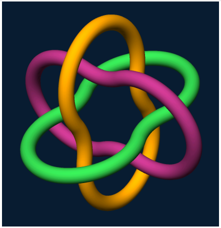
Beauty in Mathematics

Beautiful proofs

- “Proofs from the Book” (Erdős)
- share “Aha!” moment (Eureka)
- show people *why* theorem true
- make new abstract truths *visible*

Visual beauty

Optimization problems in (low-dim’l) geometry $\rightarrow$ pleasing shapes?


Four-Color Theorem

Erroneous proofs Kempe 1879 / Tait 1880

Each remained unchallenged for 11 years

Computer-assisted proof Appel/Haken 1976

- almost 2000 cases checked by computer (tour-de-force)
- not the “Proof from the Book”; hardly aids understanding

Trust this proof?

- Better than if 2000 cases checked by hand
- Computer programming very unforgiving
  while most math papers have unimportant small errors
- Now computer-verified in *Coq*:
  emphasis on *trusting* not *understanding*

Mathematical thinking styles

Many people (Felix Klein, . . .) distinguish three types

Philosoper

Conceptual

Analyst

Analytical: formulas, equations, manipulations

Geometer

Visual: pictures, diagrams, spatial relations

Try to address all three groups when teaching
R.Hamilton / R.Bryant / collaborators

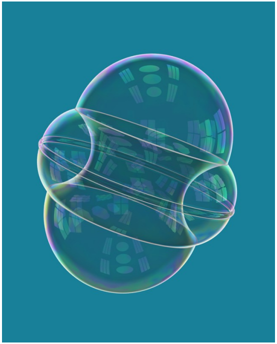
My Aha! moments – later need to work things through

Topology as source of problems in Geometry

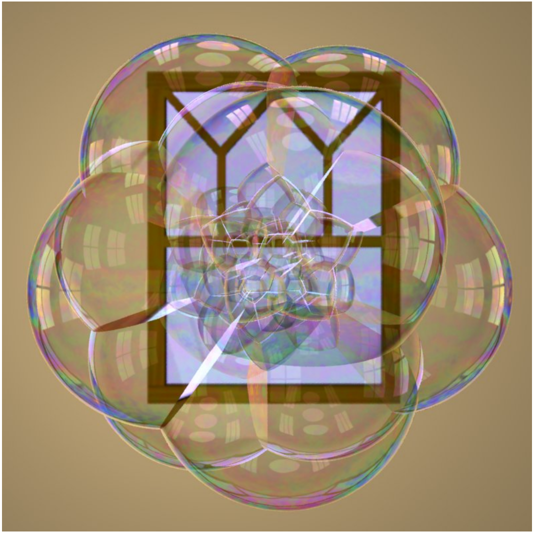
- Start with some deformable object
- Find “best” geometric representative
- Minimize some geometric energy
- Natural processes minimize free energy
- Geometric energies depend on shape
  e.g. surface tension, elastic bending energy

Example 1: Double bubble

Two soap bubbles with given volumes

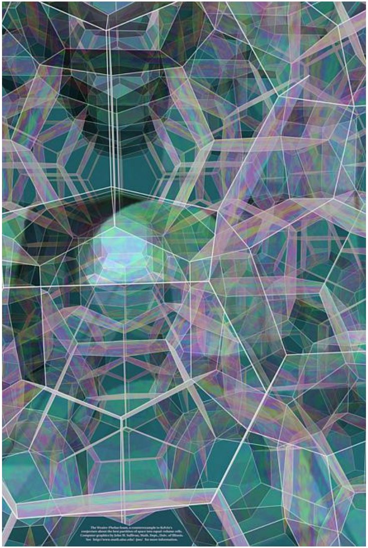


- Standard bubble best
- 2D: [Foisy 1992]; 3D, equal vol: [HS 2000]
- 3D: [HMRR 2002]; 4D: [2002]



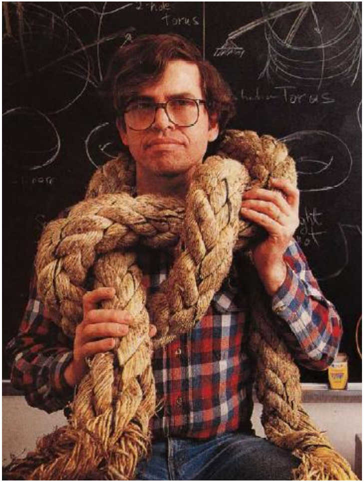
Equal-volume foams

- Partition space into
  unit-volume regions
- Kelvin [1887]
  BCC trunc. octahedra
- Weaire/Phelan [1994]
  TCP structure A15
  with two cell types

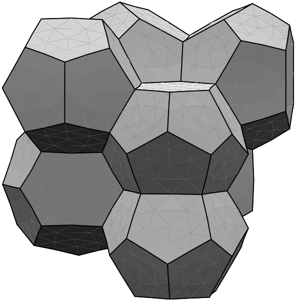


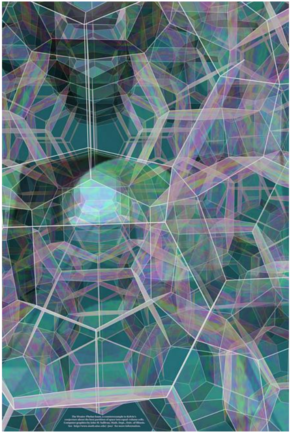
Bill Thurston (1946–2012)

- Influential vision of how to understand 3D manifolds
- Quite different to imagine a space
  - small (hold it in your hands)
  - big (live inside it)
- Wrote movingly about difficulty of expressing visions to others
- Visual insights not easily expressed in words or formulas
- Slightly easier face-to-face
- “Proof and Progress” (not “Death of Proof”)



Weaire-Phelan Foam





Tomás Saraceno: Cloud City

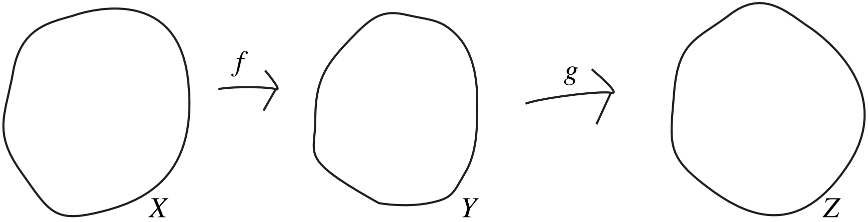
Mathematical Visualization

= using pictures to convey mathematics

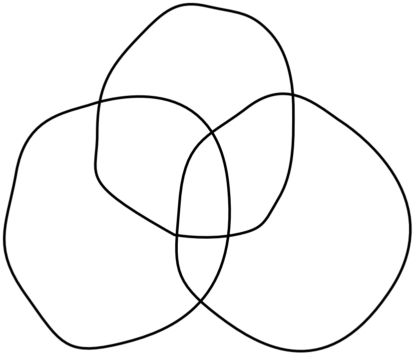
Types of pictures

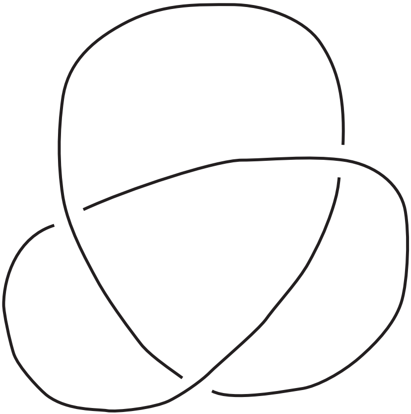
- symbolic sketches (map composition, fiber bundle)
- topological diagrams (Venn, knot, planar graph)
- proof w/o words
- 2D geometric diagram
- rendering (photorealistic?) of 3D object
- stereoscopic rendering
- 3D models / sculptures

Symbolic sketches

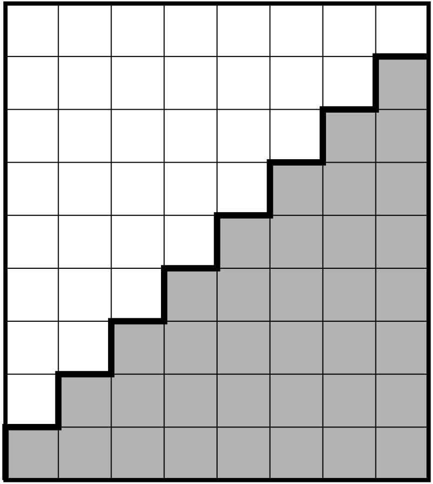


Topological diagrams



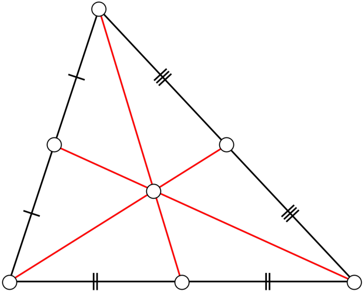


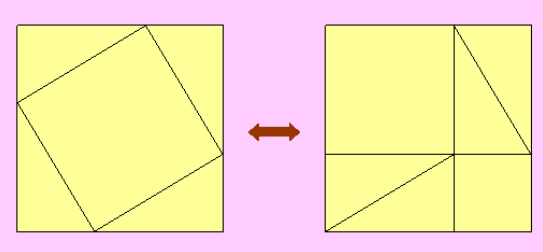
Proof without words



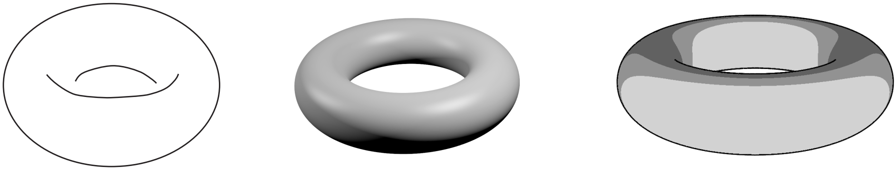
$$1 + 2 + \cdots + n = n(n+1)/2$$

2D geometric diagram

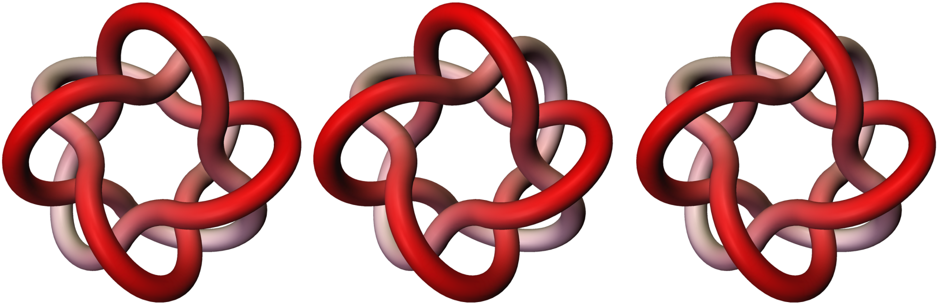




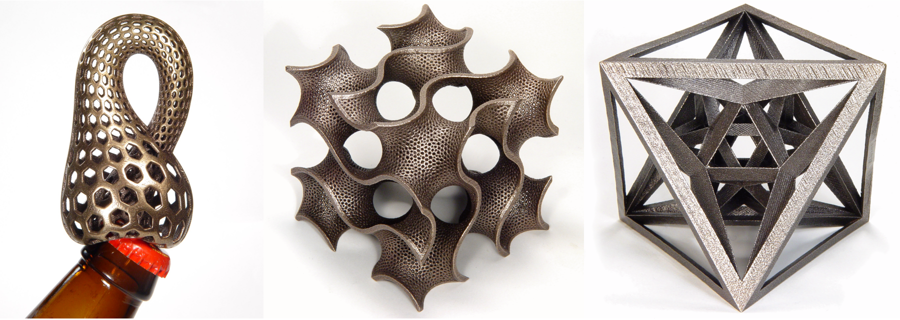
Rendering of 3D object



Stereoscopic Rendering



3D Models / Sculptures



Bathsheba Grossman

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Animations – add a time dimension

Narrative animation

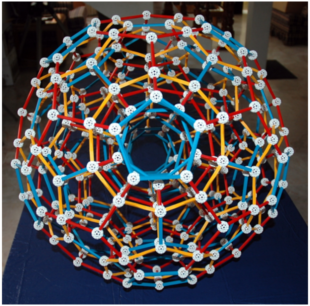
- Fixed time sequence telling a story
- Good path through higher-dimensional parameter space
- Often with voice narration
- Good for video, group presentation

Interactive animation

- User navigates through parameter space
- With guidance: limited freedom helpful
- Good for individual learning
- Now possible on all machines
- Open source (for experiments)

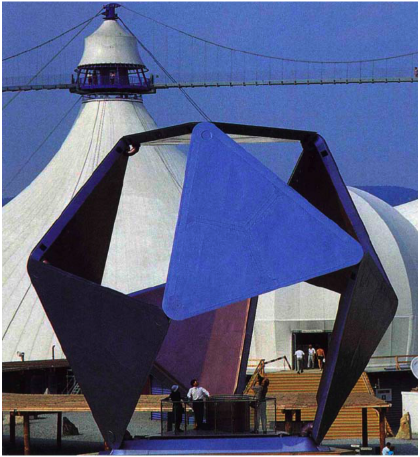
Guided interactive animation

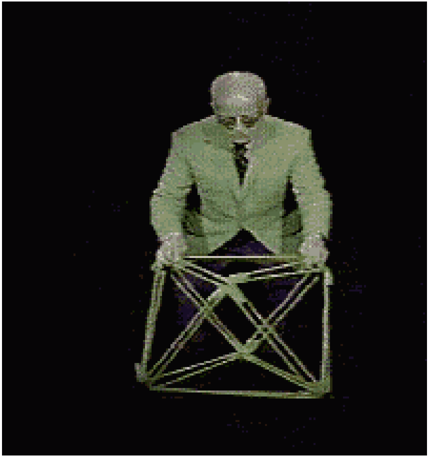
- More freedom doesn't necessarily help user
  special purpose applet say for Taylor series
- Analogous to artistic constraints helping creativity
  Sonnet form, etc., in poetry
  species counterpoint in Renaissance music (pedagogical tool)
  Arvo Pärt: tintinnabuli
- Zometool vs. more general modeling kits



Flexible models

Jitterbug with one or more degrees of freedom





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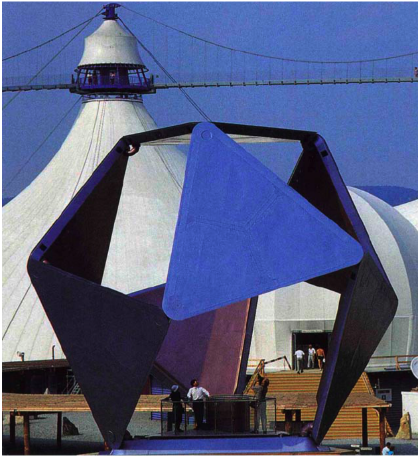

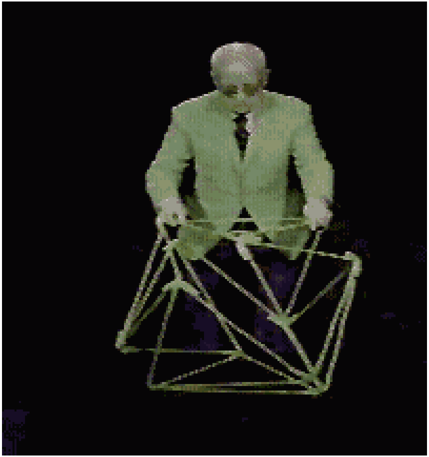

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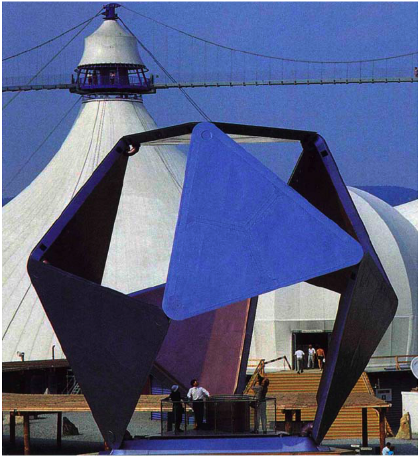

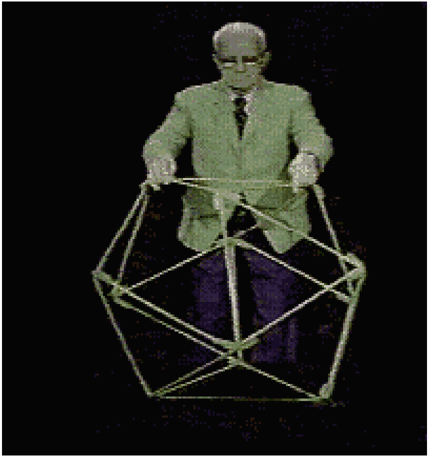

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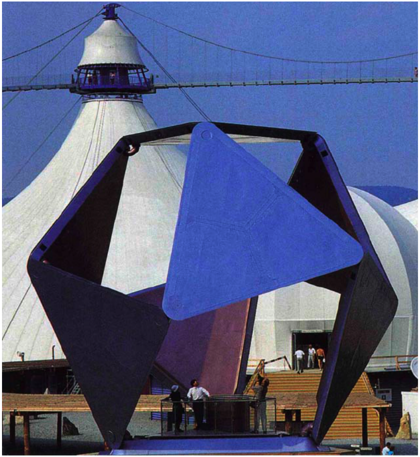
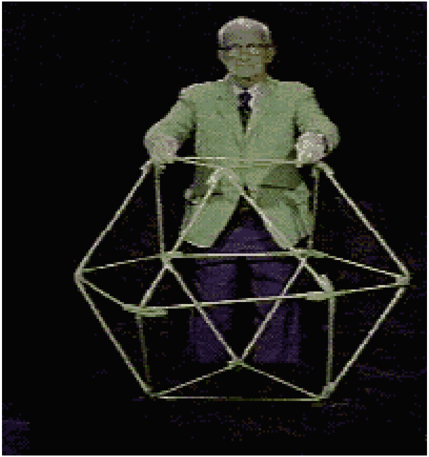
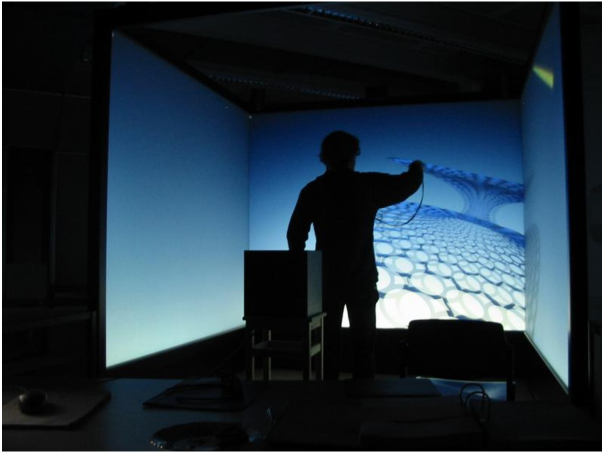




Show Loeb project

Immersive virtual reality

- Stereo, interactive, photorealistic animation, filling full visual field
- Gives user sense of being in an artificial world



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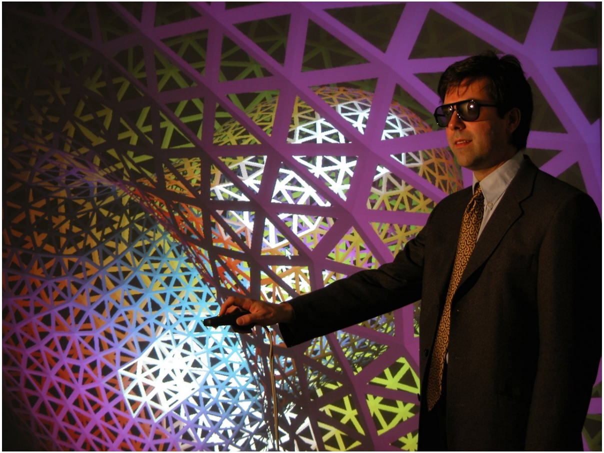


Vision and perspective

Perspective projection

- “Trivial” mathematics (matrix multiplication)
- Easy for computers
- Hard for people (except algorithmically)
  because mental model 3D

Reconstructing 3D scene

- Automatic (unconscious) for humans
- “Computer vision” very hard

Topological diagram

Easier by hand; harder by computer

Visual thinking without vision

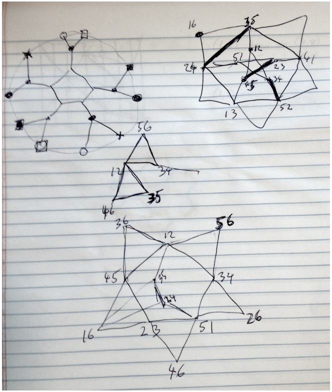
Bernard Morin

- Blind since age 5
- Expert on sphere eversions

Bill Thurston: no stereo vision

Using these pictures

- All types:
  communicating mathematics
- Computer graphics: view
  - computer experiments
  - numerical simulations
- Hand sketches:
  - work out visual ideas
  - temporary, personal meaning
  - how 3D pieces fit together
- Let K be the knot in Fig. 1 . . .
  (vs. Gauss code)

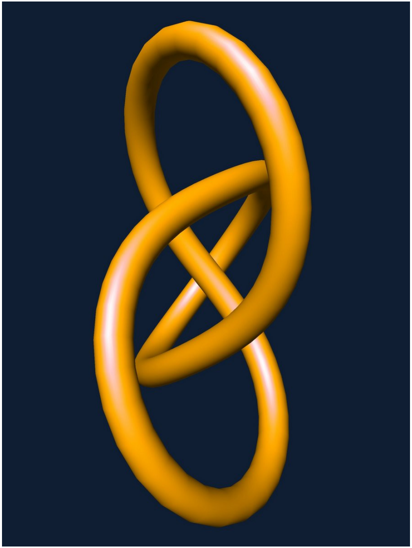


Visual imagination

- Improves with practice
- “Flatland” (Abbott, 1884)
  dimensional analogies
- “Geometry and the Imagination” Hilbert / Cohn-Vossen
  (“Anschauliche Geometrie”, 1932)

Example 2: Tight knots

- Tie a given knot in unit diameter rope
- Pull it tight (least length)
- What is its shape?
- Unknown!

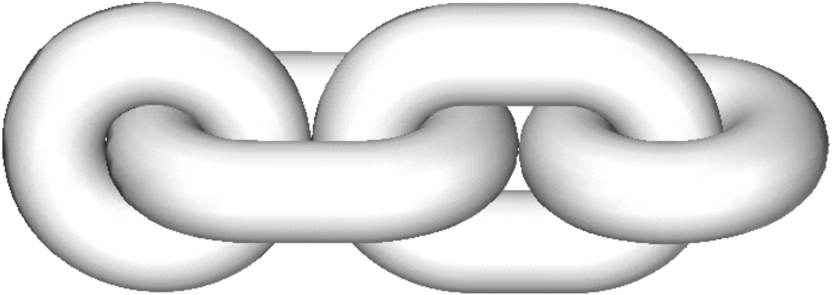


Geometric Knot Theory

- Geometric properties determined by knot type
  or implied by knottedness
- Seek optimal shape for a given knot
  (optimal geometric form for topological object)
- Minimize geometric energy

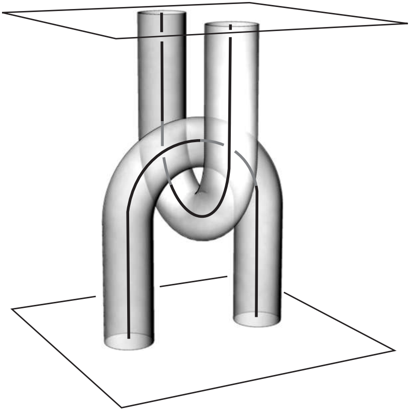
Minimizers (Tight links)

- Exist for any knot/link [CKS'02: Inventiones]
- Unknown for trefoil, figure 8, . . . any knot
- Known for some links (Proof uses minimal surfaces)
- Need not be C^2



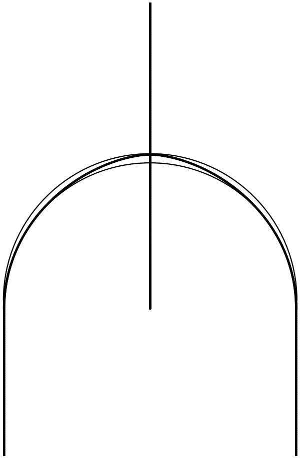
Tight clasp

- Two linked arcs
- Free boundary in $\|$ planes



Tight clasp

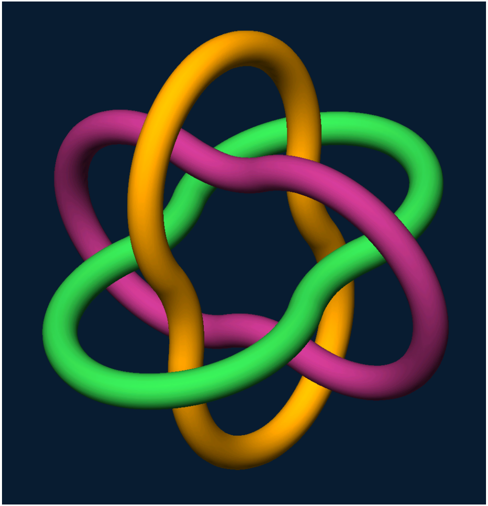
- Not semicircles!
- 0.5% shorter
- Elliptic integrals
- Curvature blows up



Borromean rings

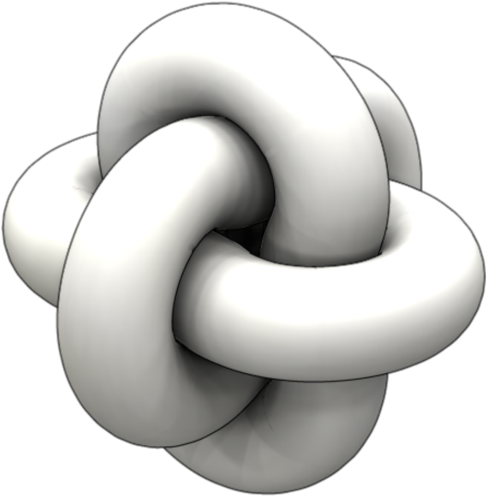
- Three linked loops
- No two are linked
- Strength in unity

[show *The Borromean Rings*]



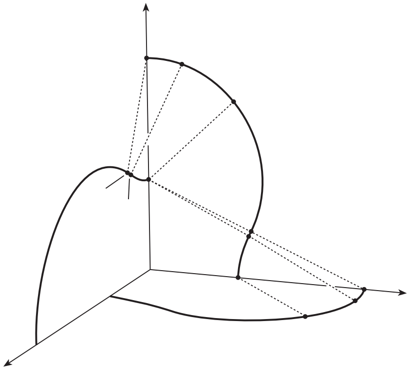
Borromean rings

- Critical configuration
- 0.1% shorter than piecewise circular

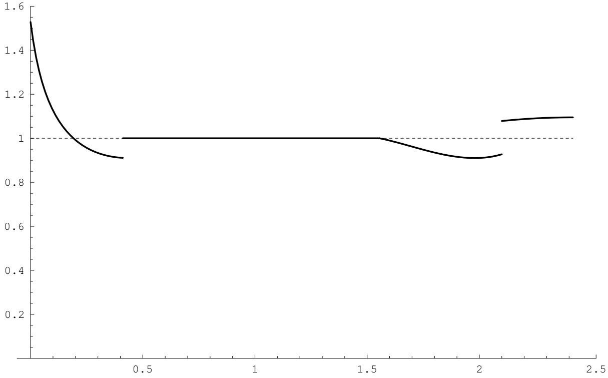


Borromean rings

- Piecewise analytic
- 42 pieces
- elliptic integrals



Curvature vs arclength



Möbius energy

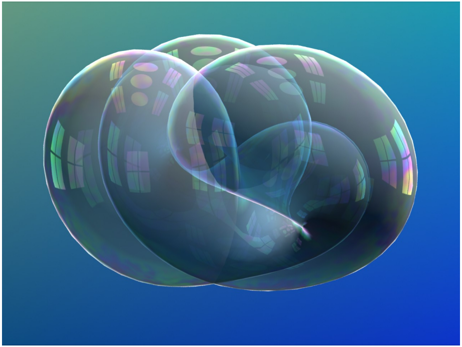
- Another notion of “best shape” for knots
- Möbius-invariant repulsive-charge energies
- Minimizers exist for prime knots [FHW]
- Some with symmetry known [KK],[KS]
- Numerical simulations
  [show video *Knot Energies*]

Willmore energy

- Surface bending energy

$$\frac{1}{4\pi} \int H^2 \, dA$$

- Cell membranes
  (lipid vesicles)



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- Surface bending energy

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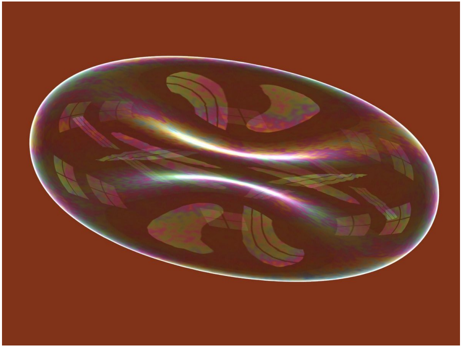
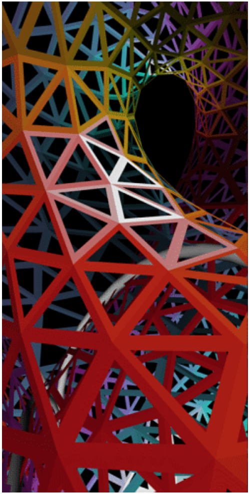


Sphere eversion

- Turn a sphere inside out
- Mathematical rules
  Not too hard (embedded)
  Not too easy (hole or crease)
- Possible [Smale 1959]
  but no explicit eversion for many years [Phillips 1966]
- Must have quadruple point [BanMax 1981]
  Simplest sequence of events [Morin 1992]
- Usually work from half-way model
- Suffices to simplify this to round sphere

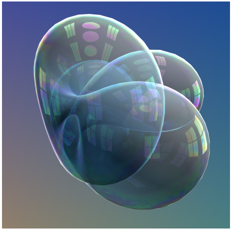
Example 3: Minimax eversion

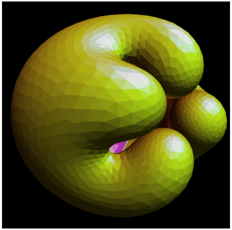
- Energy $\geq k$ for
  surface with k-tuple point
- Spheres critical for W known [Bryant]
  Lowest saddle at $W = 4$
- Use this as halfway model
  for eversion [Kusner]
- *The Optiverse*

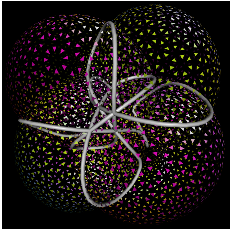


(Mathematical) Visualization Challenges

- Curved spaces, internal structure
- We usually see only outer surfaces, not inner structure
- Different depictions
  - Transparent (like soap film)
  - Solid (show shape)
  - With gaps (show self-intersections)
- Internal structure even hard to show in sculpture





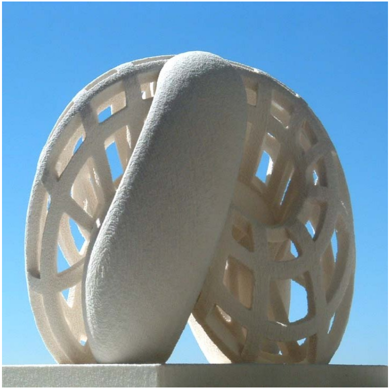


Artistic choices

- Mathematical objects have no intrinsic color (cf. Felice Frankel)
- Minimal surfaces or not?

International Snow Sculpture Championship 2004

- Our team led by Stan Wagon among 12 selected
- 20-ton, $10' \times 10' \times 12'$ block of snow
- Framework vs. solid depiction



International Snow Sculpture Championship 2004

- Our team of mathematicians among 12 selected
- 20-ton, $10' \times 10' \times 12'$ block of snow
- Framework vs. solid depiction

International Snow Sculpture Championship 2005

Symmetric sculptures

Bathsheba Grossman
Alterknot
233 (tetrahedral)

Bathsheba Grossman
Soliton
222

Symmetric sculptures

John Robinson
Genesis
$3{*}2$ (pyritohedral)

Charles Perry
Eclipse
235 (icosahedral)

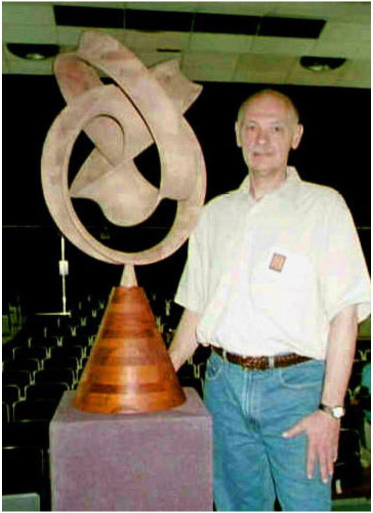
Symmetric sculptures

George Hart
Eights
235 (icosahedral)

Dick Esterle
Nobbly Wobbly
235 (icosahedral)

Brent Collins

- sculptor from Missouri
- *Visual Mind* with G. Francis
- collaboration with C. Séquin
- often $K < 0$ surfaces
  minimal?



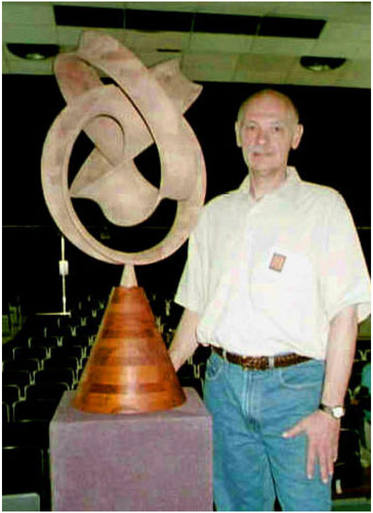
Brent Collins

Pax Mundi

Hyperbolic Hexagon II

Atomic Flower II

- wooden master
  at Bridges 1999
- merge paradigms:
  monkey saddle
  three ribbons

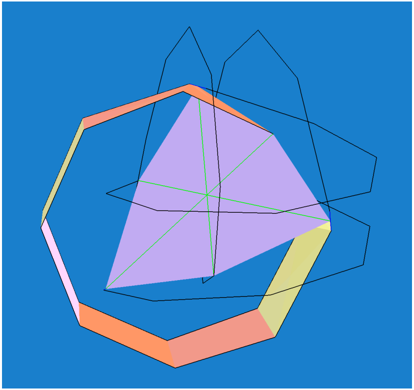


Atomic Flower II

- bronze cast 2000
  by Steve Reinmuth

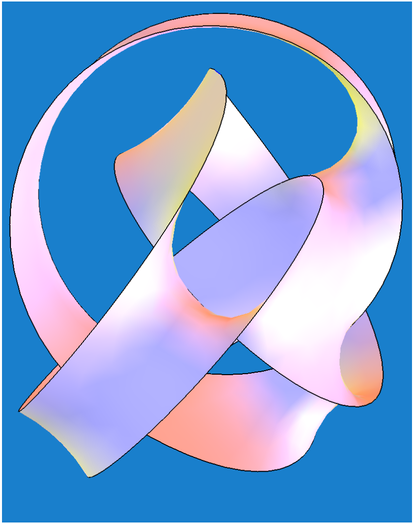
Boundary curve and initial surface

- 322 symmetry
- 3 helices, $\perp$ axes
- cubic stretch; smooth joins
- central hexagon; 3 ribons



Minimizing area

- central hexagon
  moves to one side
- 33 symmetry
- $\Rightarrow$ lines enforce 322
- ribbons insufficient curvature
- $\Rightarrow$ work in $\mathbb{H}^3$
- adjust size parameter

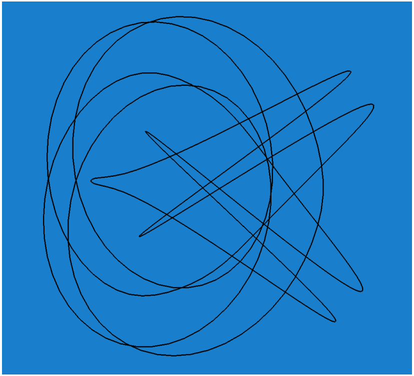


Minimal Flower 3

- not constant thickness
- instead use pressure
- CMC surfaces move too far
- homage to Brent Collins
- Intersculpt 2001
  stereolithograph

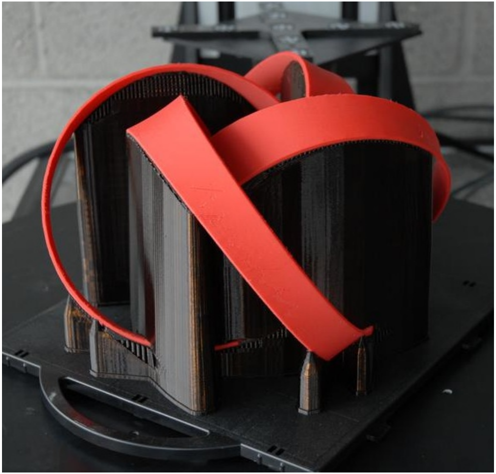
Minimal Flower 4

- 422 symmetry
- how to align 4 helices?
- same tweaks as for MF3



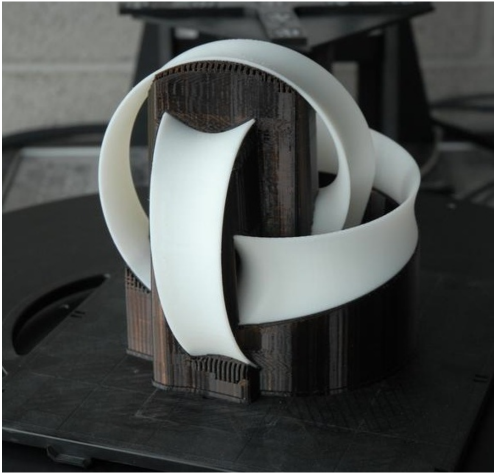
Fused Deposition Models

- support material
  chemically removed



Fused Deposition Models

- support material
  chemically removed



Minimal Flowers